SUPPORT-BASED LOWER BOUNDS FOR THE POSITIVE SEMIDEFINITE RANK OF A NONNEGATIVE MATRIX

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ABSTRACT. The postive semidefinite rank of a nonnegative $(m \times n)$-matrix S is the minimum number q such that there exist positive semidefinite $(q \times q)$-matrices $A_1, \dots, A_m$, $B_1, \dots, B_n$ such that $S(k,\ell) = \operatorname{tr} A_k^* B_\ell$. Just as the nonnegative rank characterizes the minimum size of formulations of combinatorial optimization problems as linear programs, the positive semidefinite rank characterizes their minimum size as positive semidefinite programs.

The most important lower bound technique on nonnegative rank only uses the zero/non-zero pattern of the matrix. We characterize the power of lower bounds on positive semidefinite rank based on the zero/non-zero pattern. We then use this characterization to prove lower bounds on the positive semidefinite ranks of families of matrices which arise from the Traveling Salesman and Max-Cut problems.

Keywords: Factorization rank; positive semidefinite rank; lower bounds on factorization ranks; poset embedding; combinatorial optimization.

1. INTRODUCTION

In this paper, $\Bbbk$ is a subfield of the field $\mathbb{C}$ of complex numbers. For a matrix A over $\Bbbk$, we denote its entries by $A(k,\ell)$. As usual, $A^*(k,\ell) = \overline{A(\ell,k)}$ is the Hermitian transpose, and A is positive semidefinite, if A is square, $A^* = A$, and all complex eigenvalues are nonnegative. We let $\Bbbk_+ := \Bbbk \cap \mathbb{R}_+$ denote the nonnegative numbers in $\Bbbk$. A matrix is nonnegative if all its entries are nonnegative.

Let S be an $m \times n$ nonnegative matrix over $\Bbbk$. The *nonnegative rank* of S, denoted by $\operatorname{rk}_+(S)$ is the smallest number q such that there exists a *nonnegative factorization* of S of size q, i.e., vectors $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n \in \Bbbk_+^q$ such that $S(k,\ell) = (\xi_k \mid \eta_\ell)$, where the latter is the standard inner product in $\Bbbk^q$. Similarly, the *positive semidefinite rank* of S, denoted by $\operatorname{rk}_\oplus(S)$, is the smallest number q such that there exists a *positive semidefinite factorization* of S of size q, i.e., positive semidefinite $(q \times q)$-matrices $A_1, \dots, A_m, B_1, \dots, B_n$ such that $S(k,\ell) = \operatorname{tr}(A_k^* B_\ell)$, the latter expression being the usual inner product of two square matrices. These two definitions are examples of the concept of *factorization rank*, where one wishes to write the entries of a matrix S as inner products of vectors in some Hilbert space, with diverse restrictions on the set vectors which are allowed.

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The nonnegative rank is a well-known concept in Matrix Theory, see e.g. [17, 12, 3]. Generalizations to other types of factorizations are of interest there, too, see e.g. [3, 2]. In [2], the factors ξ_k and η_ℓ are required to be in R^q, where R is some fixed semiring, e.g., a sub-semiring of $\mathbb{R}_+$. To the best of our knowledge, replacing R^q by a cone (in some inner product space over an ordered field) which is not a product of 1-dimensional cones appears to be a new concept initiated by Gouveia, Parrilo, and Thomas [10].

There is a beautiful connection between (1) factorization ranks, (2) linear mappings between convex cones, and (3) combinatorial optimization, which was first noted by Yannakakis [22] in 1991 for the nonnegative rank, and later extended by Gouveia, Parrilo, and Thomas [10]. Driven by these connections, the last several years have seen a surge of interest in factorization ranks, particularly the nonnegative rank, and recently also the positive semidefinite rank. As far as the link to combinatorial optimization is concerned, bounds—upper or lower—on the nonnegative or positive semidefinite rank provide corresponding bounds on the sizes of linear programming or semidefinite programming formulations of problems. Finding lower bounds on these factorization ranks is a difficult task, and draws on methods from combinatorial matrix theory and communication complexity.

For the nonnegative rank, the easiest, most successful, and more or less only method (but see [8]) for obtaining lower bounds just considers the support of the matrix. The *support* of S is the matrix obtained from S by replacing every non-zero entry by 1. For an $m \times n$ matrix S whose support is M, the best lower bound obtainable by considering the only the support is

$$\min\bigl\{\operatorname{rk}_+(T) \bigm| \operatorname{supp}(T) = M,\ T \geq 0\bigr\}.$$

This turns out to be equal to the *Boolean rank* of M [12], the smallest r such that there are r dimensional binary vectors $x_1, \dots, x_m \in \{0,1\}^r$ and $y_1, \dots, y_n \in \{0,1\}^r$ satisfying $M(k,\ell) = \vee_{j=1}^r x_k(j) y_\ell(j)$. The Boolean rank arises in many contexts, and is also known as *rectangle covering number* [6], *biclique covering number* [18] or, after taking $\log_2$, *nondeterministic communication complexity* [22]. Most lower bounds on nonnegative rank actually lower bound the Boolean rank, including for the recent result showing superpolynomial lower bounds on the size of linear programming formulations of the traveling salesman problem [7]. Notable exceptions to this rule include results of [22] and [14, 15].

This paper deals with the question of giving lower bounds for the positive semidefinite rank. Given the situation for nonnegative rank, it is natural to ask the following question.

Question. *How good can support-based lower bounds for positive semidefinite rank be?*

In the case of the nonnegative rank, there are plenty of examples where the Boolean rank is exponential in the rank. Moreover, it is not difficult to see that even the Boolean rank of the support of a rank-3 matrix can be unbounded [3]. In the case of the positive semidefinite rank, we will see that this is not the case: the best possible support-based lower bound for the positive semidefinite rank coincides with the minimum rank over all matrices with the same support.

Theorem 1.1. *For all 0/1-matrices M, we have*

$$\min\bigl\{\operatorname{rk}_\oplus(T) \bigm| \operatorname{supp}(T) = M,\ T \geq 0\bigr\} = \min\bigl\{\operatorname{rk}(T) \bigm| \operatorname{supp}(T) = M\bigr\}$$

The theorem answers completely the question what lower bound information can be gained about the positive semidefinite rank from the zero/non-zero pattern of a nonnegative matrix: the best possible bound is the minimum possible rank of a matrix with the given zero/non-zero pattern. De Wolf [21] calls this number the *nondeterministic rank*, and shows that the logarithm of the nondeterministic rank characterizes nondeterministic *quantum* communication complexity. We therefore have the pleasing parallel that the logarithm of the best support based lower bound for nonnegative rank is the nondeterministic communication complexity, while the logarithm of the best support based lower bound on positive semidefinite rank is the nondeterministic quantum communication complexity.

In the situation of the nonnegative rank, there is a connection between the Boolean rank and embeddings of posets: The Boolean rank of M is the minimum number of co-atoms of a truncated Boolean lattice into which a certain poset defined by M can be embedded. We prove a corresponding statement for the best-possible support-based lower bound for the positive semidefinite rank in Section 3.

Application to Combinatorial Optimization. Lower bounds of size $O(\operatorname{rk} S)$ can still be useful. The reason for this is that, whereas the nonnegative rank is lower-bounded by the rank, the positive semidefinite rank of a matrix S can be as low as[1] $\sqrt{2\operatorname{rk} S}$. Indeed, in several important situations in combinatorial optimization, the rank of the relevant matrices is $\Theta(n^2)$ (for nonnegative integers n), whereas $O(n)$ upper bounds on the positive semidefinite rank are known. The most impressive example is probably the Goemans-Williamson Positive Semidefinite Programming relaxation of Max-Cut [9].

In combinatorial optimization, one very often has a polyhedron P_0 of which one knows the vertices and extreme rays, as well as a polyhedron P_1 containing P_0, of which one knows the facets. We say that P_1 is a *relaxation* of P_0. (Typically, there are several relaxations of P_0 which are of interest in the combinatorial optimization community.) Then, in combinatorial optimization, one is faced with the following problem.

Problem. *Find a (linear or positive semidefinite) "extended formulation" for P_0 which "dominates" (i.e., is "at least as good" as) the relaxation P_1, but which is "smaller" — or prove a lower bound on the size of such an extended formulation.*

We will make the terminology clear in the next section, where we also explain the connection between extended formulation size and factorization ranks. Here, we only vaguely describe the results we obtain on two well-known combinatorial optimization problems: the Max-Cut problem and the Traveling Salesman problem.

Both problem have a parameter n which denotes the number of vertices of the graph on which the optimization problem is defined.

There is a very easy relaxation ${P_1}^\circ$ such that the Goemans-Williamson relaxation fits between the Max-Cut polytope and ${P_1}^\circ$. The Goemans-Williamson relaxation has size n, and we show that this is tight: every positive semidefinite extended formulation which dominates ${P_1}^\circ$ has size at least n. For two other well-known relaxations, the one described by the

[1]Trivially, if a positive semidefinite factorization of size q exists, the rank of S can be at most $\binom{q+1}{2}$.

so-called "triangle inequalities" and the one described by the so-called "clique inequalities", we prove that every positive semidefinite extended formulation dominating the relaxation has size $\Omega(n^2)$. The rigorous formulation of these results can be found in Theorem 2.5.

For the Traveling Salesman Problem, we prove that every positive semidefinite extended formulation dominating the well-known Dantzig-Fulkerson-Johnson relaxation (this is the one which gives the Held-Karp bound) has size $\Omega(n^2)$, see Theorem 2.8.

2. Factorizations, Polyhedra, and Combinatorial Optimization

There is a well-known connection between linear mappings between cones and factorizations of corresponding matrices. In this section, let $\Bbbk$ be a subfield of the field $\mathbb{R}$ of real numbers. Let S be a non-negative matrix, and suppose that $S = AX$ for an $(m \times d)$-matrix A and an a $(d \times n)$-matrix X, both of rank d. In other words, we are given a rank-d factorization of S. Let $Q_0 \subseteq \Bbbk^d$ be the polyhedral cone generated by the columns of X, and denote by Q_1 the polyhedral cone $\{x \in \Bbbk^d \mid Ax \geq 0\}$. Clearly, since $S \geq 0$, we have $Q_0 \subseteq Q_1$. The rank condition on A and X is equivalent to Q_0, Q_1 having dimension d.

A *linear extension* of $Q_0 \subseteq Q_1$ of size q is a polyhedral cone $\tilde{Q}$ in some $\Bbbk^s$ with q facets for which there exists a linear mapping $\pi\colon \Bbbk^s \to \Bbbk^d$ such that $Q_0 \subseteq \pi(\tilde{Q}) \subseteq Q_1$. The following is a well-known fact, going back to Yannakakis.

Theorem 2.1 ([22], c.f. [6])**.** *The minimum size of a linear extension of $Q_0 \subseteq Q_1$ equals the nonnegative rank of S.*

A *positive semidefinite extension* of $Q_0 \subseteq Q_1$ of size q is the intersection $\tilde{Q}$ of a linear subspace of some $\mathbb{M}(q \times q)$ with the set of all positive semidefinite $(q \times q)$-matrices, for which there exists a linear mapping $\pi\colon \Bbbk^s \to \Bbbk^d$ such that $Q_0 \subseteq \pi(\tilde{Q}) \subseteq Q_1$. The following fact is a straightforward generalization of a recent result by Gouveia, Parrilo, and Thomas.

Theorem 2.2 ([10], c.f. [20])**.** *The minimum size of a positive semidefinite extension of $Q_0 \subseteq Q_1$ equals the positive semidefinite rank of S.*

The situation of Theorems 2.1 and 2.2 is very common in combinatorial optimization. There, one usually starts with a polyhedron P_0 defined as the convex hull of points corresponding to feasible solutions to a combinatorial optimization problem. Finding the optimal solution would then require to optimize a linear function over P_0. For this optimization to be efficient, one would like a system of linear (or positive semidefinite) inequalities which describes P_0. However, very often, obtaining such a description is elusive. Hence, one uses a relaxation, which is a polyhedron P_1 containing P_0, defined by a set of linear inequalities satisfied by feasible solutions to the original problem. At this point, one studies so-called extended formulations. These are linear/semidefinite models of the optimization problem which may use a larger number of variables, such that the projection onto the original variable space contains P_0, and is at least as good as the relaxation P_1. One is interested in finding extended formulations with a small number of inequalities, or to prove lower bounds on the number of inequalities needed.

Since working with polyhedra that are not cones poses some unnecessary complications (among them the need to consider projective mappings π instead of linear ones), one considers Q_0, Q_1, the homogenizations of P_0, P_1. It is generally safe to assume that

- P_0 has dimension $d-1$ and
- that P_1 is pointed (i.e., it has a vertex; this is equivalent to the inequality system defining P_1 having full rank).

These two conditions imply that the rank of the slack matrix equals d. (We leave it to the reader to convice himself of that after studying Remark 2.3.) Thus, one is in the situation of the two theorems above.

For the reader who wishes to know more about the combinatorial optimization point of view, we recommend, as a starting point, the article by Kaibel [13].

Let us make precise how, from the definitions of the polyhedron P_0, generated by points and rays, and the polyhedron P_1, described by a system of linear inequalities, the slack matrix S of for the polyhedral cones $Q_0 \subseteq Q_1$ are obtained.

Remark 2.3. Let $x_1, \dots, x_{n_1}$ and $y_1, \dots, y_{n_2}$ be points in $\mathbb{k}^{d-1}$, and $a_1, \dots, a_{m'} \in \mathbb{k}^{d-1}$ and $\alpha_1, \dots, \alpha_{m'} \in \mathbb{k}$ be such that $a_k^\top x_\ell \geq \alpha_k$ for all $k = 1, \dots, m', \ell = 1, \dots, n_1$ and $a_k^\top y_\ell \geq 0$ for all $k = 1, \dots, m', \ell = 1, \dots, n_2$. In other words, the polyhedron

$$P_0 := \operatorname{cvxhull}\{x_1, \dots, x_{n_1}\} + \operatorname{cvxcone}\{y_1, \dots, y_{n_2}\}$$

is contained in the polyhedron

$$P_1 := \{x \mid a_k^\top x \geq \alpha_k \ \forall k = 1, \dots, m'\}.$$

With $n := n_1 + n_2$ and $m := 2m' + 1$, define the $(d \times n)$-matrix

$$X := \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ x_1 & & x_{n_1} & y_1 & & y_{n_2} \end{pmatrix},$$

and the $(m \times d)$-matrix

$$A := \begin{pmatrix} -\alpha_1 & a_1^\top \\ & \vdots \\ -\alpha_{m'} & a_{m'}^\top \\ 0 & a_1^\top \\ & \vdots \\ 0 & a_{m'}^\top \\ 1 & 0_{d-1} \end{pmatrix}.$$

Then the convex cone Q_0 generated by the columns of X is the homogenization of P_0, and the convex cone Q_1 defined by the system $Ax \geq 0$ is the homogenization of P_1. The entries of the slack matrix $S = AX$ can be obtained from the original data as follows. For each $k = 1, \dots, m'$ and each $\ell = 1, \dots, n_1$, we have the entry

$$S(k, \ell) = a_k^\top x_\ell - \alpha_k;$$

for each $k = 1, \dots, m'$ and each $\ell = 1, \dots, n_2$, we have the entry

$$S(m'+k, n_1+\ell) = a_k^\top y_\ell;$$

for the last row, we have

$$S(m,\ell) = \begin{cases} 1, & \text{for } \ell = 1,\dots,n_1 \\ 0, & \text{for } \ell = n_1+1,\dots,n. \end{cases}$$

2.1. **Our results on Max-Cut and TSP.**

The *Maximum Cut Problem (Max-Cut)* is the following NP-hard combinatorial optimization problem. Given a complete graph K_n on n vertices with vertex set $[n]$ and edge set $E_n := \binom{[n]}{2}$ and a weight function $c\colon E_n \to \mathbb{Q}$, find an edge cut C (i.e., inclusionwise minimal disconnecting set of edges) of K_n which maximizes the total weight

$$\sum_{e\in C} c(e).$$

Solving Max-Cut amounts to optimizing a linear function of the the famous *cut polytope*

$$P_0 := \Big\{ \chi(C) \Bigm| C \text{ edge cut in } K_n \Big\},$$

where $\chi(C)$ denotes the characteristic vector in $\Bbbk^{E_n}$ of the set C.

Many linear relaxations of the cut polytope are known; probably the best known is the one comprising the *triangle inequalities:*

$$P_1{}^{\triangle} := \Big\{ x \in \Bbbk^{E_n} \Bigm| x_{uv} \le x_{uw} + x_{wv} \quad \text{for all ordered triples } (u,v,w) \in [n]^3 \Big\}.$$

The inequalities in the following relaxation are called *clique inequalities:*

$$P_1{}^c := \Big\{ x \in \Bbbk^{E_n} \Bigm| \sum_{e\in E(U)} x_e \le {}^{|U|^2}\!/4 \quad \text{for all } U \subseteq [n] \Big\}.$$

Remark 2.4. While the Goemans-Williamson relaxation is a linear-sized positive semidefinite extension of $P_0 \subseteq P_1{}^c$, every *linear* extension of this relaxation has exponential size. (We prove this in Appendix A.)

Discarding almost all of the clique inequalities, we get the following, very simple relaxation:

$$P_1{}^\circ := \Big\{ x \in \Bbbk^{E_n} \Bigm| x_e \le 1 \quad \text{for all } e \in E_n \Big\}.$$

The clique inequalities can be strengthened to the *odd-clique inequalities* of Barahona & Mahjoub: $\sum_{e\in E(U)} x_e \le \lfloor {}^{|U|^2}\!/4 \rfloor$ for all odd-cardinality $U \subseteq [n]$. Here, for our lower bound, we require only a small number of them:

$$P_1{}^{3\text{oc}} := \Big\{ x \in \Bbbk^{E_n} \Bigm| x_{uv} + x_{vw} + x_{uw} \le 2 \quad \text{for all } u,v,w \in [n]; \Big\}.$$

We prove that the size of any semidefinite extension which dominates the nonnegativity and upper-bound inequalities (as the Goemans-Williamson relaxation) on the variables must be at least the size of the Goemans-Williamson relaxation. Moreover, adding even the

most basic inequalities (triangle or 3-odd-clique) inequalities will blow up any semidefinite extension to $\Omega(n^2)$.

Theorem 2.5. *With the above terminology:*

(a) Every semidefinite extension of $P_0 \subseteq P_1{}^{\circ}$ *has size at least* n.

(b) Every semidefinite extension of $P_0 \subseteq P_1{}^{\triangle}$ *has size at least* $(1 - o(1))\dfrac{n^2}{2}$.

(c) Every semidefinite extension of $P_0 \subseteq P_1{}^{3oc}$ *has size at least* $(1 - o(1))\dfrac{n^2}{2}$.

Remark 2.6. Item (a) is best possible since it matches the size of the Goemans-Williamson relaxation. Indeed, for this relaxation, the support-based lower bound, the trivial rank lower bound (Footnote 1) and the upper bound all coincide.

Items b and c are (asymptotically) the best possible bounds obtainable by support-based methods, since the dimension of the polytope P_0 is is $(1 - o(1))\frac{n^2}{2}$, and $P_1{}^{\circ}$, $P_1{}^{\triangle}$, and $P_1{}^{\mathrm{3oc}}$ are all pointed. Hence the rank of the slack matrix is $(1 - o(1))\frac{n^2}{2}$.

Note that both the triangle inequalities defining $P_1{}^{\triangle}$ and the 3-odd-clique inequalities defining $P_1{}^{\mathrm{3oc}}$ are special cases of the well-known *cycle inequalities*

$$x(F) - x(D \setminus F) \leq |F| - 1$$

for every (edge set of a) cyle D in K_n and $F \subseteq D$ with $|F| = 1 \bmod 2$. The cycle inequalities are arguably the most basic linear inequalities for Max-Cut. From each of (b) and (c) of Theorem 2.5, we obtain the following consequence.

Corollary 2.7. *Every semidefinite extension for Max-Cut which dominates the cycle inequalities has size at least* $(1 - o(1))\frac{n^2}{2}$.

For the *Symmetric Traveling Salesman Problem (STSP)*, given a complete graph K_n on n vertices with vertex set $[n]$ and edge set E_n and a length function $E_n \to \mathbb{Q}$, one is asked to find a Hamilton cycle in K_n of minimum total length. Solving the STSP amounts to optimizing a linear function over the *STS polytope*

$$P_0 := \left\{ \chi(C) \;\middle|\; C \text{ edge set of Hamilton cycle in } K_n \right\},$$

where, again, $\chi(C)$ denotes the characteristic vector in $\Bbbk^{E_n}$ of the set C.

Solving the STSP usually starts with the *Dantzig-Fulkerson-Johnson relaxation* [5]:

$$P_1{}^{\mathrm{dfj}} := \left\{ x \in \Bbbk^{E_n} \;\middle|\; \begin{array}{ll} \sum_{u \in [n]} x_{uv} = 2 & \text{for all } v \in [n]; \\ \sum_{e \in E(U)} x_e \geq 2 & \text{for all } U \subseteq [n], 1 < |U| \leq n/2; \\ x \geq 0 & \end{array} \right\}.$$

Optimizing over $P_1{}^{\mathrm{dfj}}$ yields the so-called *Held-Karp bound.*

There are also a few semidefinite formulations for the STSP, most notably the one by Cvetkovic et al. [4]. We prove the following.

Theorem 2.8. *Every semidefinite extension of $P_0 \subseteq P_1{}^{\text{dfj}}$ has size at least $\binom{n-2}{2}$.*

The bound in the theorem is (asymptotically) the best possible obtainable by support-based methods as the dimension of the polyhedron P_0 is $\binom{n-1}{2} - 1$, and hence the rank of the slack matrix is at most $\binom{n-1}{2}$.

3. Poset embedding ranks

In this section we give a more combinatorial interpretation of the number $\min\{\mathrm{rk}_\oplus(S) \mid \operatorname{supp} S = \operatorname{supp} M\}$.

Definition 3.1. Let S be an $(m \times n)$- matrix. We define the *poset* $\mathcal{P}(S)$ *of* S as

$$\mathcal{P}(S) := \Big(\{0\} \times \{1,\dots,m\} \cup \{1\} \times \{1,\dots,n\}\,,\ \preceq \Big),$$

where

$$(i,k) \preceq (j,\ell) \ :\Leftrightarrow\ \ i = 0 \wedge j = 1 \wedge S(k,\ell) \neq 0.$$

In other words, $\mathcal{P}(S)$ is the poset whose Hasse-diagram is the bipartite graph with lower level vertex set the row set of S and upper level vertex set the column set of S, and a vertex k of the lower level adjacent to a vertex ℓ of the upper level if and only if $S(k,\ell) = 0$.

Definition 3.2. Let $\mathcal{P}$, $\mathcal{Q}$ be posets. An *embedding* of $\mathcal{P}$ into $\mathcal{Q}$ is a mapping $j\colon \mathcal{P} \to \mathcal{Q}$ such that $x \leq y \iff j(x) \leq j(y)$ holds for all $x, y \in \mathcal{P}$.

Definition 3.3. Let S be a matrix, $\mathscr{P}$ a set of posets, and $\daleth\colon \mathscr{P} \to \mathbb{N}$. We define the $\mathscr{P}$-*embedding rank* of S as the infimum over all $\daleth(\mathcal{Q})$ such that there exists an embedding of $\mathcal{P}(S)$ into $\mathcal{Q}$.

As mentioned in the introduction, the Boolean rank of a Boolean matrix S is equal to the $\mathscr{P}$-embedding rank of $\mathcal{P}(S)$, with $\mathscr{P}$ the set of truncated Boolean lattices $\daleth(\mathcal{Q})$ the number of co-atoms of $\mathcal{Q}$ [6].

By a *subspace lattice* we mean the lattice of all linear subspaces of $\Bbbk^q$, for some $q \in \mathbb{N}$. If $\mathcal{Q}$ is the lattice of all subspaces of $\Bbbk^q$, then we let $\daleth(\mathcal{Q}) := q$. With $\mathscr{L}$ the set of all subspace lattices, it is clear that the $\mathscr{L}$-embedding rank, which we denote by $\mathrm{rk}_\star(M)$, equals the minimum dimension of a vector space in which there exist subspaces $U_1, \dots, U_m$ and $V_1, \dots, V_n$ such that

$$U_k \subseteq V_\ell \text{ if, and only if, } S(k,\ell) = 0. \tag{1}$$

In the proof of Theorem 1.1, we will prove *en passant* the following proposition.

Proposition 3.4. *For all nonnegative $(m \times n)$-matrices S, we have*

$$\mathrm{rk}_\star(S) = \min\big\{\mathrm{rk}_\oplus(T) \bigm| \operatorname{supp}(T) = \operatorname{supp}(S),\, T \geq 0\big\}.$$

More importantly,

(a) Every positive semidefinite factorization $S(k,\ell) = \operatorname{tr}(A_k^* B_\ell)$ gives rise to a subspace-lattice embedding of S of the same size by letting $U_k := \ker A_k$ and $V_\ell := \operatorname{im} B_\ell$.

(b) If $\Bbbk = \mathbb{R}$*, then in (a) we may assume that* $\dim U_k \le (\sqrt{8n+1})/2$ *and* $\operatorname{codim} V_\ell \le (\sqrt{8m+1})/2$*.*

It will become clear in the proof that, while the minimum in the subspace-lattice embeddedding rank is always attained by (co-)dimension 1 subspaces, this is not true for the subspace-lattice embedding arising from a positive semidefinite factorization.

The proposition also shows that the situation for positive semidefinite factorizations mirrors that for nonnegative factorizations. The subspace-lattice embeddedding rank is the minimum "size" $\daleth(\mathcal{Q})$ of a poset $\mathcal{Q}$ of a certain type into which $\mathcal{P}(S)$ can be embedded. The importance of such "poset embedding ranks" for factorization ranks has been noted before: it is implicit in [6] that the Boolean rank of a boolean matrix S is equal to the minimum number of co-atoms in a co-atomic poset[2] into which $\mathcal{P}(M)$ can be embedded.

4. Proof of Theorem 1.1 and Proposition 3.4

In this section we prove Theorem 1.1 and Proposition 3.4. For this, we show the following four lemmas.

Lemma 4.1. *For all nonnegative matrices* S *we have*

$$\operatorname{rk}_\oplus(S) \ge \min\bigl\{\operatorname{rk}(T) \bigm| \operatorname{supp}(T) = \operatorname{supp}(M)\bigr\}.$$

Lemma 4.2. *For all matrices* S

$$\operatorname{rk}(S) \ge \operatorname{rk}_\star(S).$$

The subspaces U_k *in the embedding can be chosen of dimension* 1*, and the subspaces* V_ℓ *of co-dimension* 1 *(and vice-versa).*

Lemma 4.3. *For all 0/1 matrices* M*, we have*

$$\operatorname{rk}_\star(M) \ge \min\bigl\{\operatorname{rk}_\oplus(T) \bigm| \operatorname{supp}(T) = M, T \ge 0\bigr\}.$$

Lemma 4.4. *Let* S *be a nonnegative matrix. Every positive semidefinite factorization* $S(k,\ell) = \operatorname{tr}(A_k^* B_\ell)$ *gives rise to a subspace-lattice embedding of* S *of the same size by letting* $U_k := \ker A_k$ *and* $V_\ell := \operatorname{im} B_\ell$*.*

Lemma 4.5. *Suppose* $\Bbbk = \mathbb{R}$*, and* S *is a nonnegative* $(m \times n)$*-matrix. If a factorization of* S *of size* q *exists, then there exists one* $S(k,\ell) = \operatorname{tr}(A_k^* B_\ell)$ *with* $\operatorname{rk} A_k \le (\sqrt{8n+1})/2$ *and* $\operatorname{rk} B_\ell \le (\sqrt{8m+1})/2$ *for all* k, ℓ*.*

Theorem 1.1 and the equation in Proposition 3.4 now follow by sticking together the inequalities. Proposition 3.4(a) follows from Lemma 4.4, and Itemb follows with Lemma 4.5.

We start with Lemma 4.1. Before we prove it, we note the following easy fact.

[2]Recall that a poset is co-atomic if every element is a meet of maximal elements. The maximal elements are then called co-atoms.

Lemma 4.6. *Suppose that $S(k,\ell) = \operatorname{tr} A_k^* B_\ell$, $k = 1,\dots,m$, $\ell = 1,\dots,n$ is a positive semidefinite factorization of S with matrices of order q. Then there exists a finite union H of proper sub-varieties of $(\Bbbk^q)^{m+n}$ such that for any $(\xi_1,\dots,\xi_m,\eta_1,\dots,\eta_n) \in (\Bbbk^q)^{m+n} \setminus H$ we have:*

$$\big(A_k\xi_k \bigm| B_\ell\eta_\ell\big) = 0 \quad\iff\quad S(k,\ell) = 0$$

In the case of $\Bbbk \in \{\mathbb{R},\mathbb{C}\}$ one can state more easily that H is a set of Lebesgue-measure zero.

Proof of Lemma 4.6. To have $(A_k\xi_k \mid B_\ell\eta_\ell) \neq 0$ for all (k,ℓ) with $S(k,\ell) \neq 0$, we need to choose (ξ,η) which do not satisfy any of the following equations:

$$(\xi_k \mid A_k B_\ell \eta_\ell) = 0; \quad (k,\ell) \text{ with } S(k,\ell) \neq 0.$$

Each of these equations defines a proper sub-variety of $(\Bbbk^q)^{m+n}$, since $0 \neq S(k,\ell) = \operatorname{tr} A_k^* B_\ell$ implies $A_k B_\ell \neq 0$. (This is most easily seen by realizing that, for $X := \sqrt{A}$, $Y := \sqrt{B}$, we have $\operatorname{tr} A^*B = \|XY\|^2$ where $\|Z\| := \operatorname{tr} Z^*Z$ refers to the Frobenius- (or Hilbert-Schmidt-) norm of the matrix Z.) □

We can now complete the proof of Lemma 4.1.

Proof of Lemma 4.1. We have to show that for every nonnegative real matrix S there exists a matrix T with $\operatorname{supp}(T) = \operatorname{supp}(S)$ and $\operatorname{rk}_\oplus S \geq \operatorname{rk} T$.

Let S be nonnegative and real with $\operatorname{rk}_\oplus S = q$, and let A_k, B_ℓ, ξ_k and η_ℓ, $k = 1,\dots,m$, $\ell = 1\dots,n$ as in Lemma 4.6. The matrix T defined by $T(k,\ell) := \big(A_k\xi_k \bigm| B_\ell\eta_\ell\big)$ has the same support as S and rank at most $q = \operatorname{rk}_\oplus S$. □

Proof of Lemma 4.2. We have to show $\operatorname{rk}_\star(S) \leq \operatorname{rk}(S)$ for all matrices S. Let $q := \operatorname{rk} S$. We give subspaces of a q-dimensional vector space W satisfying (1).

For $k = 1,\dots,m$, denote by $s_k \in \Bbbk^n$ the vector which constitutes the k-th row of S, i.e., $s_k = S(k,\dots)^\top$, and the let $U_k := \Bbbk s_k$, the linear subspace of $\Bbbk^n$ generated by s_k. The ambient space for our construction is $W := \sum_{k=1}^m U_k$, a vector space of dimension q. For $\ell \in \{1,\dots,n\}$, let K_ℓ denote the set of columns indices k with $S(k,\ell) = 0$, and define

$$V_\ell := \sum_{k\in K_\ell} U_k = \operatorname{span}\big\{s_k \bigm| S(k,\ell) = 0\big\}.$$

Clearly, $U_1,\dots,U_m$, $V_1,\dots,V_n$ are linear subspaces of a real vector space of dimension q. Moreover, by construction, we have $U_k \subseteq V_\ell$ whenever $S(k,\ell) = 0$. But since

$$V_\ell \subseteq \{x \in \Bbbk^n \mid x(k) = 0 \ \ \forall\ k \in K_\ell\},$$

we have that $S(k,\ell) \neq 0$ implies $U_k \nsubseteq V_\ell$, and we conclude (1). □

Proof of Lemma 4.3. We have to show $\operatorname{rk}_\star(M) \geq \min\big\{\operatorname{rk}_\oplus(T) \bigm| \operatorname{supp}(T) = M, T \geq 0\big\}$ for all 0/1 matrices M. For this, from subspaces of $\Bbbk^q$ satisfying (1) with S replaced by M, we constuct a matrix T and a positive semidefinite factorization with matrices of order q.

Let $U_1,\dots,U_m$, $V_1,\dots,V_n$ such a collection of subspaces. Fix any inner product of $\Bbbk^q$, and denote by A_k the matrix of the orthogonal projection of $\Bbbk^q$ onto U_k and by $B_\ell^\perp$ the

matrix of the orthogonal projection of $\Bbbk^q$ onto V_ℓ, by Id the $q \times q$ identity matrix, and let $B_\ell := \mathrm{Id} - B_\ell^\perp$. Clearly A_k and B_ℓ are positive semidefinite, and we have $A_k B_\ell = 0$ if and only if $M(k, \ell) = 0$. Thus, T defined by $T(k, \ell) := \operatorname{tr} A_k^* B_\ell$ is a matrix with $\operatorname{supp}(T) = M$, and A_k, B_ℓ a positive semidefinite factorization. □

Proof of Lemma 4.4. From a positive semidefinite factorization with matrices of order q, we will construct subspaces of $\Bbbk^q$ satisfying (1).

Let a positive semidefinite factorization of S be given, i.e., let $A_1, \dots, A_m$, $B_1, \dots, B_n$ be $q \times q$ real positive semidefinite matrices with $S(k, \ell) = \operatorname{tr} A_k^* B_\ell$. Now, for positive semidefinite matrices A, B, the two statements $\operatorname{tr} A^* B = 0$ and $AB = 0$ are equivalent. But $A_k B_\ell = 0$ is equivalent to $U_k := \operatorname{im} A_k \subseteq \ker B_\ell =: V_\ell$. □

4.1. **The case $\Bbbk = \mathbb{R}$.** For positive semidefinite matrices with real entries, the following is well-known.

Lemma 4.7 (E.g. [1]). *Let $A_1, \dots, A_m$ be square matrices, and $\alpha_1, \dots, \alpha_m$ numbers. If there exists a real positive semidefinite matrix X such that $\operatorname{tr}(A_j^* X) = \alpha_j$ for $j = 1, \dots, m$, then there exists such a matrix X with rank at most $(\sqrt{8m} + 1)/2$.*

Proof of Lemma 4.5. This lemma is an easy consequence of Lemma 4.7. We leave the easy details to the reader. □

5. Triangular rank and proofs of Theorems 2.5 and 2.8.

We start this section with a corollary to Theorem 1.1. Recall that the *trianglular rank* of a matrix is the order of the largest square submatrix which, after suitable permutation of rows and columns, is (upper or lower) diagonal, with non-zero entries along the diagonal.

Corollary 5.1. *Let S be a nonnegative matrix. The triangular rank of S is a lower bound to the positive semidefinite rank of S.*

Note that, in the case that $Q_0 = Q_1$ (i.e., S is the slack matrix of a pointed polyhedral cone), the triangular rank is trivially equal to $\operatorname{rk} S$. In [10], this fact is used to prove a lower bound for stable set polytopes.

The statement of the corollary is a trivial consequence of the second equation in Theorem 1.1. However, we give a proof based on the subspace-lattice embedding rank (the first equation in the theorem), to illustrate this method.

Proof of Corollary 5.1. W.l.o.g., we may assume that S itself is a square lower-triangular matrix of order t. Let $U_1, \dots, U_t$, $V_1, \dots, V_t$ be linear subspaces of $\Bbbk^q$ which satisfy (1). Since $S(1, 1) \neq 0$, we have $U_1 \not\subseteq V_1$, whence $U_1 \supsetneq (0)$.

Since S is lower-triangular, we have, for $k = 1, \dots, t - 1$,

$$\begin{aligned} U_1 + \dots + U_k \quad &\subseteq V_{k+1} \\ U_1 + \dots + U_k + U_{k+1} &\not\subseteq V_{k+1}, \end{aligned}$$

where the last inequality follows from $S(k+1,k+1) \neq 0$. Letting

$$W_k := \sum_{j=1}^{k} U_j$$

for $k = 0, \dots, t$ (so that $W_0 = (0)$), we conclude that, for $k = 0, \dots, t-1$,

$$W_k \subsetneq W_{k+1},$$

and it follows that $t \le q$. $\square$

We now prove Theorems 2.5 and 2.8 by giving lower bounds on the the triangular rank of the respective slack matrices.

Proof of Theorem 2.5. Let P_0 denote the cut polytope on the complete graph on n vertices, and let $P_1{}^{\circ}$, $P_1{}^{\triangle}$, $P_1{}^{\mathrm{3oc}}$ as defined on page 6.

For a vertex set $W \subseteq [n]$, denote by $\delta(W) := \{uv \in E_n \mid u \in W, v \notin W\}$ the cut defined by the bipartition $(W, [n] \setminus W)$ of the vertex set.

As mentioned in Remark 2.6, Theorem 2.5(a) can be proved either by the trivial rank lower bound or by support-based arguments. We give both proofs.

Proof of Theorem 2.5(a), rank-based version. As mentioned above, the dimension of the polyhedron P_0 is $\binom{n}{2}$ and $P_1{}^{\circ}$ is pointed, so that the dimension of the slack matrix is $\binom{n}{2}+1$. Consequently (see Footnote 1), if q is the psd-rank of the slack matrix, we have $\binom{n}{2} + 1 \le \binom{q+1}{2}$, implying $q > n-1$. $\square$

Proof of Theorem 2.5(a), support-based version. We have to prove that every semidefinite extension of $P_0 \subseteq P_1{}^{\circ}$ has size at least n. In view of Corollary 5.1, we identify, in the slack matrix S of $P_0 \subseteq P_1{}^{\circ}$, a lower triangular matrix of size n with non-zeroes along the diagonal.

The rows of the slack matrix S are indexed by edges of the complete graph; the columns are indexed by cuts. For an edge e and a cut $c = \delta(W)$ we have $S(e,c) = 0$ if and only if $e \in c$, or, in other words, $S(e,c) \neq 0$ if and only if $e \subseteq W$ or $e \cap W = \emptyset$. For ease of notation, we identify the vertex set of K_n with $\mathbb{Z}_n$. Consider the following subset of rows

$$e_j := \{j, j+1\}, \qquad \text{for } j = 0, \dots, n-2,$$

and the subset of columns

$$W_j := \{j+1\} \cup \bigcup_{i=0}^{\lfloor j/2 \rfloor} \{j-2i\}, \qquad \text{for } j = 0, \dots, n-2.$$

In words, if j is even then W_j is the set of even numbers up to and including j, and the number $j+1$; if j is odd, then W_j is the set of odd numbers up to and including j, and the

number $j+1$. For $j = n-1$ we define $e_{n-1} := \{n-1, n-1 \bmod 2\}$ and

$$W_{n-1} := \bigcup_{i=0}^{\lfloor (n-1)/2 \rfloor} \{n-1-2i\}.$$

In words, W_{n-1} is the set of numbers in $\mathbb{Z}_n$ of the same parity as $n-1$.

Clearly, we have $e_j \subseteq W_j$ for $j = 0, \ldots, n-1$. Now consider e_i and W_j for some $0 \leq i < j < n-1$. As W_j contains all numbers less than j of the same parity it will contain one of $\{i, i+1\}$; however, it will not contain the other as the only number in W_j of opposite parity is $j+1 > i+1$. In the case of W_{n-1}, it contains exactly those numbers of the same parity as $n-1$, thus will contain exactly one of $\{i, i+1\}$ for $i = 0, \ldots, n-2$.

This shows that the submatrix

$$\big(S(e_i, \delta(W_j))\big)_{\substack{i=0,\ldots,n-1,\\ j=0,\ldots,n-1}}$$

is lower triangular with non-zeros on the diagonal.

□

Obviously, the trivial rank lower bound of Footnote 1 cannot yield items (b) or (c) of Theorem 2.5.

Proof of Theorem 2.5(b). We have to prove that every semidefinite extension of $P_0 \subseteq {P_1}^{\triangle}$ has size $(1+o(1))n^2/2$. For this, we construct an lower triangular matrix of size $t := \binom{n-2}{2}$ within the slack matrix, and invoke Corollary 5.1.

The rows of the slack matrix S are indexed by triples of vertices; the triple (u, v, w) represents the inequality $x_{uv} \leq x_{uw} + x_{wv}$. The columns of S are indexed by cuts. For a triple (u, v, w) and a cut $c = \delta(W)$, we have $S((u, v, w), c) = 0$ if and only if one of the following three situations occurs:

$$u, v, w \in W \text{ or } u, v, w \notin W$$
$$\text{or}$$
$$uv \in \delta(W).$$

Now let (w_j, x_j), $j = 1, \ldots, t$, be a lexicographic enumeration of all pairs (w, x) with $1 \leq w < u \leq n-2$. For the cuts, define the sets

$$W_\ell := \{w_\ell, \ldots, x_\ell\} \cap \{\backslash\{w_\ell + 1\} \setminus \bigcup_{i=0}^{\infty} \{x_\ell - 2i - 1\}$$

For the triples, take (u_k, v_k, w_k) with and $u_k := x_k + 1$ and $v_k := x_k + 2$. We now have

$$\begin{aligned} S((u_j, v_j, w_j), \delta(W_j)) &\neq 0 \qquad && \text{for all } j = 1, \ldots, t, \text{ and} \\ S((u_k, v_k, w_k), \delta(W_\ell)) &= 0 \qquad && \text{if } 1 \leq k < \ell \leq t. \end{aligned}$$

Thus, we have a lower triangular matrix in the sense of Corollary 5.1. □

Proof of Theorem 2.5(c). We have to prove that every semidefinite extension of $P_0 \subseteq P_1{}^{3oc}$ has size $(1-o(1))n^2/2$. For this, we construct an upper triangular matrix of size $t := \binom{n-1}{2}$ within the slack matrix, and invoke Corollary 5.1.

The rows of the slack matrix S are indexed by 3-element sets of vertices; the vertex set $\{u,v,w\}$ represents the inequality $x_{uv} + x_{uw} + x_{wv} \leq 2$. The columns of S are indexed by cuts. For a vertex set $\{u,v,w\}$ and a cut $\delta(W)$, we have $S(\{u,v,w\},\delta(W)) = 0$ if and only if

$$\{u,v,w\} \cap W \neq \emptyset \text{ and } \{u,v,w\} \setminus W \neq \emptyset.$$

Let the 2-element subsets of $\{2,\dots,n\}$ be enumerated as $e_1,\dots,e_t$. Define the 3-element vertex sets $U_j := \{1\} \cup e_j$, for $j = 1,\dots,t$, and the cuts $c_j := \delta(U_j)$. Then $S(U_j, c_j) \neq 0$, for all $j = 1,\dots,t$, but $S(U_i, c_j) = 0$ for $i \neq j$. Thus, the $t \times t$ submatrix of S we have just defined is upper triangular (even diagonal) and Corollary 5.1 applies. ☐

Proof of Theorem 2.8. Let P_0 denote the Traveling Salesman polytope on the complete graph on n vertices, and let $P_1{}^{\mathrm{dfj}}$ denote the Dantzig-Fulkerson-Johnson relaxation as defined on page 7.

Proof of Theorem 2.8. We have to prove that every semidefinite extension of $P_0 \subseteq P_1{}^{\mathrm{dfj}}$ has size $\binom{n-2}{2}$. For this, we exhibit an upper triangular matrix of size $\binom{n-2}{2}$ within the slack matrix, and invoke Corollary 5.1.

We will need only the rows of the slack matrix S which are indexed by edges $e \in E_n$, corresponding to the inequalities $x_e \geq 0$. The columns of the slack matrix S are indexed by Hamilton cycles C, and we have $S(e,C) = 0$ if and only if $e \notin C$.

To simplify notation, we identify the vertex set of K_n with $\mathbb{Z}_n$.

Let $R \subset \mathbb{Z}_n \times \mathbb{Z}_n$ be defined as $R = \{(k,\ell) : k,\ell \in \mathbb{Z}_n, k < \ell - 1, \ell < n-1\}$. Let $r := |R| = \binom{n-2}{2}$. The starting point for our construction is the Hamilton cycle

$$C_0 := \Big\{\{0,1\},\{1,2\},\dots,\{n-2,n-1\},\{n-1,0\}\Big\}$$

in K_n. For each $(k,\ell) \in R$, the edge set

$$C_{k,\ell} := \Big(C_0 \setminus \Big\{\{k,k+1\},\{\ell,\ell+1\}\Big\}\Big) \cup \Big\{\{k,\ell\},\{k+1,\ell+1\}\Big\}$$

forms a Hamilton cycle in K_n. Note that all $C_{k,\ell}$ are distinct. Consider the $r \times r$ submatrix T of S whose rows correspond to the edges $e_{k,\ell} := \{k,\ell\}$, $(k,\ell) \in R$ (listed in lexicographical order), and whose columns correspond to the Hamilton cycles $C_{k,\ell}$, $(k,\ell) \in R$. We have $S(e_{k,\ell}, C_{k',\ell'}) \neq 0$ if and only if $(k,\ell) = (k',\ell')$ or $(k,\ell) = (k'+1,\ell'+1)$. As $(k'+1,\ell'+1)$ occurs after (k',ℓ') in lexographical order, T will be lower triangular with non-zero diagonal entries. Thus, the matrix T is a triangular submatrix whose diagonal entries are non-zero, whence, by Corollary 5.1, we conclude that $\mathrm{rk}_{\oplus} S \geq r = \binom{n-2}{2}$. ☐

6. Outlook

As we have shown, support-based lower bounds on the positive semidefinite rank of a matrix will always be at most the rank. (In fact, one might wonder whether the rank

of a matrix is always an upper bound on its positive semidefinite rank, but for each $r \geq 3$, Corollary 4.16 in [10] gives families of matrices with rank r and unbounded positive semidefinite rank.) We illustrate how lower bounds which move beyond considering the support might be based on subspace-lattice embeddings via Proposition 3.4.

Example 6.1. With $\Bbbk := \mathbb{R}$, consider the $(n \times n)$-matrix S_n where $S_n(i,j) = (i-j-1)(i-j-2)/2$. We have $\operatorname{rk} S_n = 3$ for all n, which follows from the expansion

$$(i-j-1)(i-j-2) = (i^2 - 3i + 1) + (j^2 + 3j + 1) - (2ij),$$

as each term in parenthesis can be expressed as a rank one matrix.

We conjecture that the positive semidefinite rank of S_n grows unboundedly with n. (Note that the bound in [10, Corollary 4.16] does not apply since S_n is not the slack-matrix of a polytope.) We can prove the following.

Claim. If $n \geq 6$, the positive semidefinite rank of S_n is at least 4.

Proof of the claim. By considering the upper-left 6×6 submatrix, it suffices to prove the claim for $n = 6$:

$$S_6 = \begin{bmatrix} 1 & 3 & 6 & 10 & 15 & 21 \\ 0 & 1 & 3 & 6 & 10 & 15 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 1 & 0 & 0 & 1 & 3 & 6 \\ 3 & 1 & 0 & 0 & 1 & 3 \\ 6 & 3 & 1 & 0 & 0 & 1 \end{bmatrix}$$

By contradiction, assume that $A_1, \dots, A_6$, $B_1, \dots, B_6$ is a positive semidefinite factorization of S_6 of order 3.

Let U_k, V_ℓ be subspaces of $\mathbb{R}^3$ as in Proposition 3.4. Since for $k \geq 3$, the kth row contains zeros and non-zeros, we have $\dim U_k \geq 1$ for these k. For the same reason, we have $\dim V_\ell \leq 2$ for $\ell \leq 4$. If we had $\dim U_k = 2$ for any $k \geq 3$, then, for ℓ, ℓ' with $S_6(k,\ell) = S_6(k,\ell') = 0$, it would follow that $V_\ell = V_{\ell'}$, which is impossible since the ℓth column differs from the ℓ'th. Thus we conclude that $\dim U_k = 1$ for $k \geq 3$. Similarly, we have $\dim V_\ell = 2$ for $\ell \leq 4$.

But this means that A_k, $k \geq 3$, and B_ℓ, $\ell \leq 4$, are rank-1 matrices. Choose vectors $u_k, v_\ell \in \mathbb{R}^3$, $k = 3, \dots, 6$, $\ell = 1, \dots, 4$, such that $A_k = u_k u_k^\top$, and $B_\ell = v_\ell v_\ell^\top$. For these k, ℓ, we have

$$S_6(k,\ell) = \operatorname{tr}(u_k u_k^\top v_\ell v_\ell^\top) = (u_k^\top v_\ell)^2 = Y(k,\ell)^2,$$

where we define the rank-3 matrix $Y(k,\ell) := u_k^\top v_\ell$. Since $Y(k,\ell) = \pm\sqrt{S_6(k,\ell)}$, we may enumerate all the 2^9 possible choices for Y. Doing this, we see that all possible choices for Y have rank at least 4, so no such Y can exist, a contradiction. (We note that, independently, the technique based on entry-wise square roots has been used and further developed in [11].) □

This example shows how using additional structure of a positive semidefinite factorization—for example that if S has a rank-one semidefinite factorization of dimension k then there is a

matrix Y of rank k whose entrywise square is S—can lead to improved lower bounds. The following concrete problems motivate finding more general methods that can show positive semidefinite rank lower bounds larger than the rank.

As stated in Corollary 2.7, the slack matrix of the important cycle-relaxation of Max-Cut has positive semidefinite rank $\Omega(n^2)$. The Boolean rank is $O(n^4)$, and no non-trivial lower-bound is known for the nonnegative rank. (For the nonnegative rank, as pointed out in the introduction, the rank is a trivial lower bound.) For both the nonnegative and the positive semidefinite rank, lower bounds of $\Omega(n^{2+\varepsilon})$ for some $\varepsilon > 0$ would be very interesting.

For the slack matrix of the Dantzig-Fulkerson-Johnson relaxation of the Traveling Salesman Problem, an upper bound on the nonnegative rank of $O(n^3)$ is known. It would be interesting to improve the lower bounds for both the nonnegative and the positive semidefinite rank (both are $\Omega(n^2)$). It would be especially interesting to known whether these two ranks are within a factor of $o(n)$ of each other in this situation.

On a more theoretical side, one might ask whether, for a real matrix S, the positive semidefinite rank over $\Bbbk := \mathbb{R}$ can be larger than the positive semidefinite rank over $\Bbbk := \mathbb{C}$. This mirrors the corresponding problem posed by Cohen & Rothblum [3, Section 5] (cf. [2]) regarding the nonnegative rank over the reals of rational matrices.

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Appendix A. Linear extensions of the cut polytope dominating the clique inequalities

Here we prove Remark 2.4. Let P_0 denote the cut polytope of a complete graph on n vertices, and let P_1^c as defined on page 6.

We quickly repeat some definitions from graph theory. Let G be a bipartite graph with vertex set $X \amalg Y$ (here, and in what follows, $\amalg$ stands for the disjoint union of two not necessarily disjoint sets). A *biclique* in G is a complete bipartite graph with vertex set $U \amalg V$, for sets $U \subseteq X$, $V \subseteq Y$, which is a subgraph of G. We say that biclique B *covers* an edge $(u, v) \in E(G)$, if $(u, v) \in E(B)$. The *biclique covering number* is the minimum number of bicliques in G required to cover all edges of G.

As mentioned in the introduction, the biclique covering number is just the Boolean rank of the Boolean $X \times Y$-matrix M with $M(x, y) = 1$ if and only if $(x, y) \in E(G)$. It is a lower bound to the nonnegative rank of any matrix S with $\operatorname{supp} S = M$.

To lower bound the biclique covering number, we will use a lower bound on the communication complexity of the *unique disjointness* problem. For strings x, y of length N, the unique disjointness function evaluates to 1 if $x \cap y = \emptyset$ and evaluates to 0 if $|x \cap y| = 1$. Otherwise, it is undefined. The specific result we will need is an exponential lower bound

on the number of bicliques to cover the ones of unique disjointness, provided that no biclique covers a zero of the function. This lower bound follows simply from the key lemma of Razborov [19] used to prove an $\Omega(n)$ lower bound on the randomized communication complexity of disjointness, and is explicitly stated in [21]. For a textbook treatment, see Lemma 4.49 in Kushilevitz & Nisan [16].

We now more precisely state the exact result we need. Let $N = 4\ell + 1$ be an odd integer. Denoting the set of all ℓ-element subsets of $\{1, \dots, N\}$ by $\binom{[N]}{\ell}$, let H_N denote the bipartite graph with vertex set $\binom{[N]}{\ell} \amalg \binom{[N]}{\ell}$, and $(x, y) \in E(H_N)$ if and only if $x \cap y = \emptyset$. Let $\bar{H}_N$ denote the bipartite graph on the same vertex set and with $(x, y) \in E(\bar{H}_N)$ if and only if $|x \cap y| = 1$ (so $E(\bar{H}_N) \cap E(H_N) = \emptyset$). A biclique is called $\bar{H}_N$*-feasible*, if it contains no edge of $\bar{H}_N$. (It may, however, contain edges which are neither in H_N nor in $\bar{H}_N$, i.e., it might not be a biclique in H_N.)

Theorem A.1 ([19, 21])**.** *The number of $\bar{H}_N$-feasible bicliques needed to cover all edges in H_N is $2^{\Omega(N)}$.*

This theorem was used in [7] to prove their main result. Our usage is different, though.

Proof of Remark 2.4. By Theorem 2.1, we have to prove that the slack matrix for the relaxation $P_0 \subset P_1^c$ has nonnegative rank $2^{\Omega(n)}$. The approach is to bound the Boolean rank. For this, we will give a lower bound to the biclique covering number of the following bipartite graph G. The vertices of G are, on the one hand, the cliques U, $U \subset V_n$, and on the other hand the cuts $\delta(W)$, $W \subset V_n$. We have $(U, \delta(W)) \in E(G)$ if and only if $|U \cap \delta(W)| < {}^{|U|^2}\!/_4$. In other words, there is *no* edge between U and $\delta(W)$, if and only if $|U \cap W| = |U|/2$.

We make no attempt to optimize the constant in the exponent.

Assuming that $n = 2 \bmod 8$, let $N := n/2$, $\ell := \lfloor N/4 \rfloor$. Given a list of r bicliques in G which cover every edge of G, we construct a covering of the edges of H_N by (at most) r bicliques which are $\bar{H}_N$-feasible. Consider the sets U with cardinality $2\ell - 2$ which have $U \cap \{N+1, \dots, n\} = \{N+1, \dots, N+\ell-2\}$, and the sets W of cardinality u which satisfy $U \cap \{N+1, \dots, n\} = \{N+1, \dots, N+\ell-2\}$, too. Then $|U \cap W| = {}^{|U|}\!/_2 = \ell - 1$ if and only if the two ℓ-element sets $x := U \cap \{1, \dots, N\}$ and $y := W \cap \{1, \dots, N\}$ satisfy $|x \cap y| = 1$. Thus, a covering of the the edges of the subgraph of G spanned by this type of vertices U and $\delta(W)$ of G by bicliques in G gives a covering of the edges of H_N by $\bar{H}_N$-feasible bicliques. By Theorem A.1, such a covering must have $2^{\Omega(n)}$ bicliques. $\square$

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